Algebraic curves
Solutions sheet 9

May 8, 2024

Unless otherwise specified, k is an algebraically closed field.

Exercise 1. Let R be a ring and $I, J \subseteq R$ two ideals. I, J are said to be *comaximal* if $I + J = R$.

1. Show that $IJ \subseteq I \cap J$ for any I, J and that equality holds for comaximal ideals. Can you provide a counter-example when I, J are not assumed to be comaximal?

2. Suppose I, J are comaximal. Show that, for $m, n \geq 1$ I^m, J^n are also comaximal.

Consider now ideals $I_1, \ldots, I_N \subseteq R$. For $1 \leq i \leq n$, call $J_i = \bigcap_{j \neq i} I_j$.

3. Suppose that for all i, I_i, J_i are comaximal. Show that for all $n \geq 1$, $I_1^n \cap \ldots \cap I_N^n = (I_1 \ldots I_N)^n = (I_1 \cap \ldots \cap I_N)^n$.

Finally, consider the k-algebra $R = k[x_1, \ldots, x_n]$ and ideals $I, J \subseteq R$.

4. Show that I, J are comaximal if, and only if, $V(I) \cap V(J) = \emptyset$.

Solution 1.

1. $\subseteq$. Let $a \in IJ$. Then there are $i \in I$, $j \in I$ such that $a = ij$. From this $a \in I \cap J$.

   $\supseteq$. Suppose I, J comaximal. Let $a \in I \cap J$. $a = a \cdot 1 = a(i + j) = ai + ja \in IJ$.

   Counterexample. $R = k[t]$. $I = J = (t)$. $IJ = (t^2) \not\supseteq I \cap J = (t)$.

2. Let $1 = i + j$. Suppose $m \geq n$. Then $I^m + J^n \subseteq I^n + J^n$. Suffices to show it for $m = n$. Then $1 = (i + j)^{2m} = \sum_{k=0}^{2m} c_k^{2m} i^{2m-k} j^k$. For $k \leq m$, $i^{2m-k} \in I^m$ and for $k \geq m$ $j^k \in J^m$. Hence $I^m + J^m = R$.

3. Proof by induction on N. By induction, $I_1^n \cap \ldots \cap I_{N-1}^n = J_N^n$. So $I_1^n \cap \ldots \cap I_N^n = J_N^n \cap I_n^n$. By comaximality of I_N^n and J_N^n, $I_1^n \cap \ldots \cap I_N^n = J_N^n I_n^n = (J_N I_n)^n$. By comaximality of J_N and I_n, get $I_1^n \cap \ldots \cap I_N^n = (J_N \cap I_n)^n$.

4. Suppose that I, J are comaximal and $x \in V(I) \cap V(J) = V(I + J)$. It means that for $f \equiv 1 \in k[x_1, \ldots, x_n]$, $1 = f(x) = 0$, which is a contradiction.

Exercise 2. Let R be a ring. Recall that a domain is called *integrally closed* if, for any $x \in K = Frac(R)$, if there exist $a_1, \ldots, a_n \in R$ such that $x^n + a_1 x^{n-1} + \ldots + a_n = 0$, then $x \in R$. Show that R is a DVR if, and only if, R is an integrally closed Noetherian local domain with Krull dimension one. (Hint: You can use without proof that any ideal $I \neq (0), R$ in a Noetherian, dimension 1, integrally closed domain can be written uniquely as a product of prime ideals. Can you find a uniformizer of R?)

Solution 2.

- $DVR \Rightarrow$ integrally closed Noetherian local domain with Krull dimension 1.

  Suppose R is a DVR. By definition, it is a Noetherian local domain. It has Krull dimension 1 because all ideals are (0) or (t^n) where (t) is the unique maximal ideal. Only $(0) \not\supseteq (t)$ are prime. PID implies UFD which implies integrally closed.

- $DVR \Leftarrow$ integrally closed Noetherian local domain with Krull dimension 1.

  Suppose R is integrally closed Noetherian local domain with Krull dimension 1. Take $\mathfrak{m}$ unique maximal. Using the hint with $I = \mathfrak{m}^2$, $\mathfrak{m}^2 \not\supseteq \mathfrak{m}$. Take $\omega \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then using the hint with $I = (\omega)$, we get $\mathfrak{m} = (\omega)$.

Exercise 3. A *valuation* on a field K is a surjective function $\varphi : K \to \mathbb{Z} \cup \{\infty\}$ satisfying the following axioms:

(i) $\varphi(a) = \infty \Leftrightarrow a = 0$

(ii) $\varphi(ab) = \varphi(a) + \varphi(b)$

(iii) $\varphi(a + b) \geq min(\varphi(a), \varphi(b))$

Show that the datum of a DVR with quotient field K is equivalent to the datum of a valuation on K i.e.

1. Given a valuation φ on K, $R = \{\varphi \geq 0\}$ is a DVR with maximal ideal $\mathfrak{m} = \{\varphi > 0\}$.
2. Given a DVR R, ord is a valuation on R (assuming we set $ord(0) = \infty$).

Now consider $K = \mathbb{Q}$ and $p \in \mathbb{Z}$ some prime number.

3. Show that $\mathbb{Z}_{(p)}$ is a DVR. What is the associated valuation ord_p?
4. Show that any valuation on $\mathbb{Q}$ is equal to ord_p for some prime number p. (Hint: Using Bezout's theorem, you can show that a valuation on $\mathbb{Q}$ is strictly positive in at most one prime.)

Solution 3.

1. Let $R = \{\varphi \geq 0\}$ and $\mathfrak{m} = \{\varphi > 0\}$ is an ideal.

   R is local : let $x \in R \setminus \mathfrak{m}$. $\phi(x) = 0$. Then $\phi(x^{-1}) = -\phi(x) = 0$ so $x^{-1} \in R$. Hence $x \in R^{\times}$.

   A uniformizer is given by an element of valuation 1.

2. Let $a, b \in R^*$. Let t be a uniformizer. Then $a = ut^n$, $b = vt^m$. $ab = uvt^{n+m}$, so $ord(ab) = ord(a) + ord(b)$. $a + b = ut^n + vt^m$. Suppose $n \leq m$. Clearly $ord(a + b) \geq n = min\{ord(a), ord(b)\}$.

3. $\mathbb{Z}_{(p)} = \{r \in \mathbb{Q} \mid r = \frac{a}{b}, b \notin p\mathbb{Z}\}$. A uniformizer is given by p. $ord_p(r)$ is just $v_p(a)$.

4. Let v be a valuation on $\mathbb{Q}$.

   - $v(\mathbb{Z}) \subseteq \mathbb{Z}_{\geq 0}$
   - $v(p_1^{n_1} \cdot \dots \cdot p_r^{n_r}) = n_1 v(p_1) + \dots + n_r v(p_r)$. $v(x) = 1 \Rightarrow x$ is prime.
   - $p \neq q$ primes. By Bézout $up + vq = 1$, $u, v \in \mathbb{Z}$, so $0 = v(1) \geq min\{v(up), v(wq))\}$. Hence at most one of $v(p)$ or $v(q)$ is 1.

     As v is surjective, take the unique p such that $v(p) = 1$. Then $v = v_p$. Indeed $v(q) = 0$ for all $q \neq p$.

Exercise 4. A simple point P on a curve F with tangent line L at P is called a *flex* if $ord_P^F(L) \geq 3$. The flex is called *ordinary* if $ord_P^F(L) = 3$ and a *higher* flex otherwise.

1. Let $F = Y - X^n$. For which n does F have a flex at $P = (0,0)$ and what kind of flex?

2. Suppose that $P = (0,0)$, $L = Y$ and $F = Y + aX^2 + \dots$ (the remaining terms having degree at least 2). Show that P is a flex if, and only if, $a = 0$. Give a simple criterion for calculating $ord_P^F(Y)$.

Solution 4.

1. $F = Y - X^n$. $P = (0,0)$. The tangent line is $L = Y$. X is a uniformizer of $\mathcal{O}_P(F)$. $ord_P^F(Y) = n$ because $Y = X^n$. Hence F has a flex of order n for $n \geq 3$.

2. As before, X is a uniformizer of $\mathcal{O}_P(F)$. $P = (0,0)$. The tangent line is $L = Y$. In $\mathcal{O}_P(F)$, $Y = -aX^2 + \dots$ hence has valuation ≤ 2 if and only if $a \neq 0$.

   Simple criterion : if we write $F = Y(1 + F_1(X,Y)) + X^m F_2(X)$ with $X \nmid F_2(X)$, $F_1(0,0) = 0$. In $ord_P^F(Y)$,

$$Y = -X^m(1 + F_1(X,Y))^{-1} F_2(X)$$

   Hence $ord_P^F(Y) = m$.

Exercise 5. Let $V = V(X^2 - Y^3, Y^2 - Z^3) \subseteq \mathbb{A}_k^3$, $P = (0,0,0)$ and $\mathfrak{m} = \mathfrak{m}_P(V)$. Compute $dim_k(\mathfrak{m}/\mathfrak{m}^2)$.

Solution 5.

$$\mathfrak{m}/\mathfrak{m}^2 = (x,y,z) \otimes_{k[x,y,z]} \mathcal{O}_P(V)/\mathfrak{m}$$

$$\mathfrak{m}/\mathfrak{m}^2 = k \cdot x \oplus k \cdot y \oplus k \cdot z$$

so $dim_k(\mathfrak{m}/\mathfrak{m}^2) = 3$.